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Stochastic Mechanics and Quantum Theory

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Stochastic mechanics may be regarded as both generalizing classical mechanics to processes with intrinsic randomness, as well as providing the sort of detailed description of microscopic events declared impossible under the traditional interpretation of quantum mechanics. It avoids the many conceptual difficulties which arise from the assumption that quantum mechanics, i.e., the wave function, provides a complete description of (microscopic) physical reality. Stochastic mechanics presents a unified treatment of the microscopic and macroscopic domains, in which the process of measurement plays no special physical role and which reduces to Newtonian mechanics in the macroscopic limit.

KEY WORDS: Quantum paradoxes; collapse of the wave packet; measurement process; uncertainty principle; complementarity; wave–particle duality; hidden variables; nonlocality.

1. INTRODUCTION

Consider an electron in the ground state of the hydrogen atom. We have all been taught that we should not ask for a detailed description of the behavior of the electron beyond what is provided by the ground-state wave function. If this is so, it must be because the electron is not the sort of pointlike entity that can be regarded as always having a definite position. Of course, we should not be too surprised if concepts of a sort completely different from those appropriate for the description of macroscopic reality are required for the microscopic realm, which is so remote from our immediate experience. However, some explanation is required of why, whenever we try to locate an electron, we find it at a well-defined position, as if the measurement process itself somehow converts our electron into an

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object that, for a short time at least, is the sort of thing that does have a definite position. Though it is natural to ask how this comes about, we are often told that this, too, should not be asked. But the inadmissability of this question cannot arise from the electron's unfamiliarity—it is reasonable to ask of any object, whether familiar or not, why it behaves as it does—but rather from its being no object at all. That is, we must believe that in microscopic physics we do not describe an objective reality, which has an existence independent of the observer, but rather we describe relations between our observations—between, that is, subjective mental states. But then how objective can macroscopic reality be, constructed, as it is, out of the microscopic?

If we do persist in asking, as most of us are inclined to do, how, by observing an electron, we produce an object at a definite position, we presumably answer the question by treating the process of measurement quantum mechanically. But then we find the situation becoming much worse than we bargained for: The quantum mechanical treatment of the measurement process transfers our original difficulty about the sort of thing an electron is to the level of macroscopic objects. We find that the "pointer" of our measuring apparatus is, after the measurement, in a quantum state described by a superposition, the terms of which correspond to different positions of our pointer. It thus appears that macroscopic objects, too, are not the sort of things that we thought they were; and it is, of course, much more difficult to accept this of familiar macroscopic reality than of electrons.

But even if we accept this of macroscopic objects, our problems are not over, because the dynamics of quantum mechanics still gives us no hint as to how the statistical outcomes of the measurement process arise. When we perform repeated identical measurements on a system, always in the same initial state, we typically get different answers; the statistical distribution of these answers is consistent with the probability interpretation of the wave function. But it does not appear to be consistent with the quantum mechanical time evolution provided by, say, Schrödinger's equation, which is completely deterministic. It thus appears, as Bohr[1] has emphasized, that quantum mechanics should not be regarded as providing a description encompassing the macroscopic as well as the microscopic realm. (It is true that the many-worlds interpretation[2] of quantum mechanics claims to reconcile quantum dynamics with quantum probabilities, but in addition to being extremely bizarre, it does not appear to succeed.)

I wish to describe a theory, stochastic mechanics, which answers all the questions just raised, and many more; and does so in a way that is so simple and natural that the first reaction of the reader, after understanding the argument, should be that there must be a mistake somewhere, that it cannot all be so simple. As of yet, I have found no mistake.

Stochastic mechanics was discovered by Fényes[3] and developed and put into its modern form by Nelson.[4–6] The version of stochastic mechanics I wish to describe here is mathematically Nelson's, but conceptually perhaps somewhat different. Nelson regards stochastic mechanics as arising from a deeper, purely classical theory, while I wish to suggest that we consider stochastic mechanics as standing on its own; that we suppose that stochastic mechanics is *the* theory, and see where this leaves us. When we do this we find that stochastic mechanics retains most of the valuable insights associated with quantum mechanics, while avoiding aspects that are unpalatable, if not incoherent. We see how and why the observer plays a critical role, and why composite systems must often be regarded as somehow defining an indivisible whole. We avoid such embarrassments as the abandonment of objective reality, the introduction of a fundamental role to be played by consciousness,[7] the problem of the collapse of the wave packet, and the need to posit an inescapable separation between a *classical* observer and a quantum system. Stochastic mechanics provides a detailed description of how the uncertainty principle arises, not in terms of the abandoned theory, as in Heisenberg's explanation using classical mechanics, but rather solely in terms of the new theory itself. Moreover, it encompasses both the macroscopic and microscopic realms, and in the macroscopic limit reduces to classical mechanics. We also see, and this is directly built into stochastic mechanics, how probabilities arise, and we get a clear picture of the status of the collapse of the wave packet. Finally, we can see how, without directly trying to do so, the limitations set by Bell's inequalities are naturally overcome.

In what follows I hope to explain what stochastic mechanics is, and how it accomplishes all I have claimed for it. I begin, in Section 2, with a very brief description of quantum mechanics. In Section 3, I discuss some of the familiar paradoxes arising from the assumption that the wave function provides a complete description of the phenomenon under investigation. The discussion of stochastic mechanics begins in Section 4 with a brief description of the theory, and continues in Sections 5 and 6 with a consideration of the two-slit experiment in stochastic mechanics and a comparison of the derivations of quantum and stochastic mechanics. The key section is Section 7, which the measurement process is analyzed for stochastic mechanics. I briefly discuss the uncertainty principle from the standpoint of stochastic mechanics in Section 8. Nonlocality is considered in Section 9, while Section 10 concerns topological effects. Section 11 is devoted to some additional remarks. Finally, in Section 12 I present some concluding remarks, closing with some conservative implications.

For simplicity, the discussion will be limited to the physics of the nonrelativistic Schrödinger equation, and, for the most part, the Pauli exclusion principle will be ignored.

2. QUANTUM MECHANICS

In quantum mechanics, with a system of N particles we associate the Hilbert space $\mathscr{H} = L^2(\mathbb{R}^{3N})$. A state of our system is described by the wave function, a vector $\psi \in \mathscr{H}$. The dynamics is given by the unitary evolution $U_t: \mathscr{H} \to \mathscr{H}$, $\psi_0 \to \psi_t$, where $U = \exp[-i(t/\hbar) H]$, with $H = -(\hbar^2/2m) \Delta + V$, if the particles have mass m and interact via the potential V. This evolution is, of course, deterministic.

The predictions of quantum mechanics are determined by the following statement:

(∗) $|\psi(x)|^2 \, dx =$ the probability of finding the configuration of the system in dx about $x \in \mathbb{R}^{3N}$.

To the extent that every measurement is ultimately a position measurement, (∗) suffices for all predictions.

3. THE PARADOXES

The familiar paradoxes associated with quantum mechanics arise from the assumption, implicit in the traditional interpretations of quantum mechanics, that ψ provides a complete description of the system, omitting no facts.

Let us consider the measurement process. Here we regard our system together with the measuring apparatus as a composite system, to be treated quantum mechanically: System $\otimes$ Apparatus = Composite. If the initial system state is ψ and the initial apparatus state is ϕ, then for the initial state of the composite system we have

$$\Phi_i = \psi \otimes \phi$$

The quantum mechanical evolution, when applied to this initial state, leads to a final state of the form

$$\Phi_f = \sum \psi_\alpha \otimes \phi_\alpha$$

in which the system is correlated with the apparatus. For example, for $N = 1$, suppose

$$\psi = (\psi_l + \psi_r)/\sqrt{2}$$

is a superposition of a wave function ψ_l supported to the left of the origin and ψ_r supported to the right. If we want to determine whether our particle is on the left or the right, we use an apparatus such that

$$\begin{aligned}\Phi_f &= [(\psi_l \otimes \phi_l) + (\psi_r \otimes \phi_r)]/\sqrt{2} \\ &= (\Phi_l + \Phi_r)/\sqrt{2}\end{aligned}$$

Here, we may think of the apparatus as having three relevant states: ϕ, in which the "pointer" points straight up; ϕ_l, in which it points to the left; and ϕ_r, in which it points to the right.

Note that Φ_f is a superposition, a pure state, not a mixture; while after the measurement we expect (and get) a definite result, either l or r. How does this come about? It is customary at this point to invoke a new principle, incompatible with the unitary evolution of quantum mechanics, namely "the collapse of the wave packet." We are thus faced with the problem of reconciling this principle with the assumed completeness of the Φ_f description.

The usual sorts of answers one finds are the following:

1. The measurement process must be treated classically.[1]

The natural response to this is: Why? And how is this consistent with the completeness assumption?

2. In order to learn whether the apparatus reads l or r, we must make a further observation.

The problem with this approach is that it leads to an infinite regress, or to the many-worlds interpretation of quantum mechanics, neither of which is particularly congenial.

Here is another problem with 2: It is not pleasant to think of macroscopic objects as being in schizophrenic states. In fact, we may replace our apparatus with a cat and arrange that

$$l \leftrightarrow \text{dead}, \qquad r \leftrightarrow \text{alive}$$

and obtain the paradox of Schrödinger's cat.[8] Or we may allow our apparatus to be a person, with

$$l \leftrightarrow \text{person read } l, \qquad r \leftrightarrow \text{person read } r$$

and obtain the paradox of Wigner's friend[7] (see also Ref. 9).

3. We may replace Φ_f by the obvious mixture (density matrix) because they are observationally equivalent, since, to all intents and purposes, interference is impossible between ϕ_l and ϕ_r.[10] (The basic difference

between a mixture and a superposition is that the possibility of interference is lost in the mixture.) In one version of this response, one takes the thermodynamic limit and argues that in this limit the pure state in fact converges weakly to a mixture.[11]

The problem with this answer is that our composite system, however large it may be, is not infinite, and hence is in a pure state. (It is important for the reader to understand the difference between the role of the thermodynamic limit here and in statistical mechanics.) The main problem, the completeness assumption, has not been addressed, for the following question has not been answered: How has the evident fact that after the measurement the pointer is in the definite state of pointing, say, to the left been created?

4. STOCHASTIC MECHANICS

I will first review classical mechanics very briefly. In classical mechanics, the state of our system of N particles is described by a point $(x, p) \in \Gamma = \mathbb{R}^{3N} \times \mathbb{R}^{3N}$, the phase space, and the time evolution

$$\Gamma \to \Gamma, \qquad (x_0, p_0) \to (x_t, p_t)$$

is deterministic, and in fact is given by ordinary differential equations.

Stochastic mechanics is a sort of mixture of quantum mechanics, classical mechanics, and diffusion theory. The state space in stochastic mechanics for our system of N particles is

$$\mathbb{R}^{3N} \times \mathscr{H}$$

where the $\mathbb{R}^{3N}$ should be thought of as the configuration space of classical mechanics and $\mathscr{H}$ as the quantum Hilbert space. The state is thus given by

$$(x, \psi), \qquad x \in \mathbb{R}^{3N}, \quad \psi \in \mathscr{H}$$

The time evolution is given by the autonomous evolution of $\psi \in \mathscr{H}$, which biases the evolution of $x \in \mathbb{R}^{3N}$:

$$\psi_0 \to \psi_t = U_t \psi \qquad \text{(quantum evolution)}$$

and

$$dx_t = b\, dt + (\hbar/m)^{1/2}\, dW_t \tag{4.1}$$

where W_t is standard Brownian motion (on $\mathbb{R}^{3N}$), so that x_t undergoes a diffusion process with drift b, which is related to ψ as follows:

$$b \equiv b(x, t) \equiv b(\psi) = \frac{\hbar}{m} \nabla(R + S) \tag{4.2}$$

where $\psi = \exp(R + iS)$. [If the ith particle has mass m_i, then m in the above equations should be regarded as the matrix, diagonal(m_i), having m_i in the diagonal slots allotted to the ith particle.]

A key consequence of this choice of b is that if the probability density $\rho_t(x)$ of this diffusion process at some time t satisfies

$$(*) \qquad \rho_t(x) = |\psi_t(x)|^2$$

then $(*)$ is satisfied for all later times. It follows, in view of the sufficiency of position measurements, that stochastic mechanics yields the same predictions as quantum mechanics, provided $(*)$ is satisfied (for some past t).

But why should $(*)$ be satisfied? Can it be regarded as an additional constraint in the theory? If stochastic mechanics is to be regarded as standing on its own, and not as arising from some deeper theory, it makes no sense to regard $(*)$ as an additional constraint, since $\rho(x)$ is a classical probability density and ψ is a dynamical element of the theory.

It can be shown, however, that whatever ρ_0 is,

$$\|\rho_t - |\psi_t|^2\| \equiv \int |\rho_t(x) - |\psi_t(x)|^2|\, dx$$

is decreasing in time t, and in some cases it can be shown that it decreases to 0, in fact very quickly, so that $(*)$ is effectively satisfied after a short time. It is also easy to find examples where this is not so; but it seems that, in fact, this does not matter, and that for all practical purposes we may assume $(*)$. I do not wish to go into details here, but this assertion rests on the following observations:

1. Since $\|\rho_t - |\psi_t|^2\|$ is decreasing, if at any time in the past $(*)$ is (approximately) satisfied, it will be (approximately) satisfied at all later times. Consider, for example, an electron in an excited state of the hydrogen atom. Then $\|\rho_t - |\psi_t|^2\|$ need not decrease to 0, since (it can be shown that) the electron cannot diffuse across nodal surfaces (where $\psi = 0$). But as soon as we ask how the electron managed to get itself into the excited state, we realize that it was presumably by excitation from the ground state, in which $\|\rho_t - |\psi_t|^2\|$ quickly decreases to 0.

2. As will be explained in Section 7, the measurement process provides us with information on the basis of which we should update ρ; the new probability density $\hat{\rho}$ so obtained will usually not satisfy $(*)$. But, as will also be explained, we are also justified in updating ψ, in effect collapsing the wave packet to $\hat{\psi}$, and if $(*)$ were satisfied before the measurement, it will also be satisfied after the measurement, by $\hat{\rho}$ and $\hat{\psi}$. Moreover, even if $(*)$ were not satisfied before the measurement, $\|\hat{\rho} - |\hat{\psi}|^2\|$ will, on the average, be smaller than $\|\rho_t - |\psi_t|^2\|$ before the measurement.

Thus we have a second mechanism, the measurement process, which helps in bringing $(*)$ about.

The relationship between ψ and the diffusion process x_t can be summarized by saying that ψ is a "pilot wave" which guides the motion of the coordinates x in such a way that x wants to be where $|\psi|$ is large and tends to avoid regions where $|\psi|$ is small.

5. THE TWO-SLIT EXPERIMENT

Stochastic mechanics thus provides a simple answer to the question that naturally arises in connection with the two-slit experiment: How does the electron, which, after all, passes through only one of the slits, know whether both slits are open, so that the arrival of electrons on a photographic plate behind the slits is described by an interference pattern, or whether only one slit is open, so that there is no interference? Whatever the electron may know, the wave function ψ with which it is associated, and which together with x provides a complete description of the electron in stochastic mechanics, does know: when both slits are open, ψ has the familiar interference profile, which guides the electron, i.e., the coordinates x, along channels to produce the pattern, while if one slit is closed, there is neither profile nor channels.

6. DERIVATIONS

I wish here to compare briefly derivations of quantum and stochastic mechanics.

In one derivation of quantum mechanics, the Poisson bracket $\{\ ,\ \}$ of classical mechanics is replaced by the commutator. In Schrödinger's derivation, the energy and momentum are replaced by suitable differential operators in the energy equation. I mention these to emphasize how formal these derivations are.

To derive stochastic mechanics, consider the class of general $\hbar$ diffusions, i.e., processes satisfying (4.1) with a general drift b, not necessarily satisfying (4.2). Then stochastic mechanics [i.e., $\hbar$ diffusions satisfying (4.2) for some solution ψ of Schrödinger's equation], with $\rho_t(x) = |\psi_t(x)|^2$, is what arises from the stochastic version of:

(i) Force = mass × acceleration.[4,5]

Or

(ii) The Lagrangian variational principle.[12,13]

Thus, stochastic mechanics does not arise merely in a formal way, but as a natural extension of classical mechanics to processes with "intrinsic randomness."

7. THE MEASUREMENT PROCESS

Let $\Phi_i = \psi \otimes \phi$ be the initial wave function for the composite of system and apparatus, as in Section 3. Then in stochastic mechanics the complete initial state is given by

$$((x, y), \psi \otimes \phi)$$

where y describes the apparatus coordinate. In the example of Section 3 involving ψ_l and ψ_r, on which we shall, for simplicity, focus here, the initial value of y is the initial position of the pointer, namely $\uparrow$. The possible final (postmeasurement) states, which arise from the diffusion of (x, y) under Φ, are of the form

$$(x, l; \Phi_f) \quad \text{if} \quad x \text{ is on the left } (x \in l)$$

$$(x, r; \Phi_f) \quad \text{if} \quad x \text{ is on the right } (x \in r)$$

where, for simplicity, we assume that the apparatus has but two final positions, l and r. (It would be more accurate to regard l and r as describing ranges of values for y.)

That $(x \in l,\ y = l)$ and $(x \in r,\ y = r)$ are the only possibilities already follows from the structure of Φ_f [and the assumption (*) of Section 4]; it is not necessary to analyze the evolution in detail. We should also arrange, by proper choice of apparatus, that x does not change (much) during the measurement, if we wish to regard the result of the procedure as conveying clear information about the position x of the particle before the measurement.

Moreover, collapse of the wave packet, i.e., replacing, say, Φ_f by Φ_l when we find that $y \in l$, is justified by the observation that the evolution of (x_t, y_t) beginning at $(x \in l,\ y = l)$ under Φ_l should be the same as under Φ_f. Φ_f is a *disjoint superposition*, $\Phi_f = (\Phi_l + \Phi_r)/\sqrt{2}$, with Φ_l and Φ_r having disjoint supports, the terms of which correspond to different readings of the apparatus. Moreover, since Φ_l and Φ_r correspond to incompatible macroscopic situations, the superposition should remain (approximately) disjoint at all later times. (It might be useful here to keep macroscopic irreversibility in mind, and/or to think of $y = l$ as a written record.) From this follows the (near) impossibility of the coordinate—(x, y)—diffusion crossing the "boundaries" of the supports, i.e., passing from the support of

Φ_l to the support of Φ_r. Thus, if $y = l$ immediately after the measurement, the coordinates (x_t, y_t) will remain in the support of Φ_l, where the drift b knows (essentially) nothing of Φ_r, for all later times.

In other words, explanation 3 of the paradoxes (see Section 3)—that collapse is justified because of the practical impossibility of interference[10] (see also Ref. 14)—more or less works in stochastic mechanics, where ψ alone is not a complete description, nor even, from a macroscopic perspective, the most important part of the description. In stochastic mechanics, the fact that $y = l$ is not created by collapse. The stochastic mechanical description already embodies this fact prior to collapse; or, more precisely, regardless of whether, for the sake of convenience, we choose to collapse.

Since this is an important point, let me, at the risk of belaboring, restate it. The Φ result of a (quantum) measurement is a disjoint superposition, the terms of which correspond to different readings of the measuring device. More or less by explanation 3 of the paradoxes, according to which the superposition can be replaced by a mixture after the measurement, the superposition will remain forever (approximately) disjoint. Corresponding to the disjoint superposition of Φ, we obtain a disjoint decomposition of the drift b, as well as the impossibility of the diffusion crossing supports. Hence, collapse, if desired, is justified.

Though the analysis given above was for a position measurement of a particularly simple kind, any position measurement can be treated in a similar manner. Thus, from the perspective of stochastic mechanics, there is nothing particularly strange about position measurements. These roughly follow the lines of a "classical" measurement prescription. We measure the position of a (microscopic) particle by allowing it to interact with a device which, without (much) affecting the particle's position, establishes a strong correlation between this position and that of a macroscopic object, so that by looking at the latter we (more or less) know the former.

There are, nonetheless, fundamental differences between purely "classical measurement" and measurement in stochastic mechanics. First of all, in stochastic mechanics, as in quantum mechanics, for measurements other than of position, e.g., momentum measurements, the measurement process can usually not be regarded as revealing something that already exists, but rather, at best, must be regarded as creating what did not exist prior to the measurement. Moreover, if one takes into account its effect upon the wave function of the system under observation, then even for position measurements the measurement process cannot generally be regarded as leaving the system essentially undisturbed. While I do not wish to go very deeply into these matters, a brief consideration of the uncertainty principle in stochastic mechanics might be appropriate at this point.

8. THE UNCERTAINTY PRINCIPLE

Diffusing particles do not have a well-defined instantaneous velocity. Thus, an aspect of the uncertainty principle is built into the structure of stochastic mechanics. But one can do much better than this.

Shucker[15] has shown how to define the momentum as a random variable on path space in stochastic mechanics. Not determined by (x, ψ), the momentum corresponds, roughly speaking, to the long-time asymptotic average velocity that would arise if the initial wave function were allowed to evolve freely. Thus defined, the momentum has the distribution given by quantum mechanics—hence, the uncertainty principle.

Moreover, one has in stochastic mechanics a detailed description of how, in obtaining sharp position information, we must destroy sharpness in momentum. In fact, the states of sharply defined momentum correspond to wave packets with sharp wave vector. The result of a precise position measurement will be, after collapsing, as discussed above, an approximate δ-function, i.e., a broad superposition of wave packets with a wide range of wave vectors, a state that even in stochastic mechanics does not have a well-defined momentum; a state about which there is no fact of the matter as to the value of the momentum, since this is simply not embodied in the (x, ψ) description resulting from the measurement. On the other hand, a momentum measurement would produce a wave packet, embodying a (fairly) well-defined momentum, where none existed prior to the measurement; this would lead, of course, to a loss of position information if the position variance were small before the measurement. But it should be noted again that while the momentum measurement leads to a destruction of position *information*, the position itself is not destroyed: the position is embodied in (x, ψ). (A more detailed analysis of the relationship between position and momentum uncertainties arising from measurement in stochastic mechanics would be in accordance with the Heisenberg uncertainty relation $\Delta p\, \Delta q \geqslant \hbar/2$.)

9. NONLOCALITY

Stochastic mechanics is nonlocal: it involves instantaneous effects between components of a system that do not diminish as the distance between the components grows. This can easily be seen directly—the wave function is, after all, a nonlocal entity. But I prefer to argue as follows.

Quantum mechanics, as well as any theory whose predictions agree with those of quantum mechanics, is nonlocal: Consider a widely separated pair of spin-1/2 particles (particle 1 and particle 2) in the singlet ($s = 0$) state. Once I measure any spin component of particle 1, I also know the same component for particle 2. Either this component for particle 2 (or, more precisely, the fact that this component has the value we now know it to have) did not exist before the measurement, in which case we have a stark violation of locality, since a distant action would have caused something to pop into existence instantaneously; or the component, and hence all spin components, had well-defined values, i.e., were "elements of reality,"[16] before the measurement. In the latter case we say we have "hidden variables" (HV). What I have just described is a reformulation of the famous Einstein, Podosky, Rosen (EPR) argument.[16,17] I summarize:

(i) EPR: locality + quantum mechanics $\Rightarrow$ HV

But Bell's inequality,[18] which must be satisfied by any local HV theory for the spin components of our particles, is violated by quantum mechanics:

(ii) Bell: locality + quantum mechanics $\Rightarrow$ no HV

The obvious conclusion of (i) and (ii) is that quantum mechanics, or any theory whose predictions agree with those of quantum mechanics, is nonlocal. Moreover, since Aspect's experiments[19] verify the quantum mechanical predictions violating Bell's inequalities for spin correlations, nature is nonlocal.

Thus, since its predictions agree with those of quantum mechanics, stochastic mechanics (at least the extension of what I have described which incorporates spin; see Refs. 6 and 20) must be nonlocal. And, as stated above, this can easily be directly checked even for stochastic mechanics without spin, which I have been discussing. This is indeed a good thing for stochastic mechanics, if, as I have argued, nature itself is nonlocal.

(It might be instructive for the reader at this point to employ the analyses of Sections 7 and 8 to see how the EPR phenomenon, in its original form involving just the positions and momenta of two particles,[16] appears in stochastic mechanics.)

The nonlocality argument is usually presented differently: It is observed that since stochastic mechanics can be regarded as a HV theory, whose predictions agree with those of quantum mechanics, it follows from Bell's inequality that it must be nonlocal. I believe that, while correct, this argument, by apparently blaming hidden variables for nonlocality, lets quantum mechanics off too easily.

Though the version of stochastic mechanics I have discussed is nonrelativistic, the nonlocality it embodies, since it does not diminish with distance, would presumably also be present in a relativistic version. But how would this nonlocality then be compatible with Lorentz invariance? To get a handle on this question, we must look more carefully at the nature of the nonlocality occurring in stochastic mechanics (or, for that matter, in quantum mechanics).

The most problematic sort of nonlocality occurring in stochastic mechanics is the following: Consider two widely separated components of a composite system. Then the marginal distribution on trajectories for one of the components, i.e., the marginal *process* for the coordinates of that component, will in general be more or less immediately affected by a "measurement" performed on the other component. Different measurements on one component will lead to different marginals for the other component. (Note that I have said nothing about the *conditional* process for one component, given the *result* of the measurement on the other component.) The marginal *single-time* distributions will not, however, be affected by the measurement on the other component; according to quantum mechanics, they cannot be: the reduced density matrix for one component is not affected by measurements performed on the other component.

Now if it were possible to observe the trajectories of one component without disturbing them, so that the statistics of observed trajectories agree with the predictions of stochastic mechanics, then, using many copies of our two-component system, it would be possible to transmit information with arbitrarily large velocity, which is incompatible with special relativity. But such observations cannot be performed in stochastic mechanics: as soon as an observation is performed at one time, the wave function is affected and future statistics are altered. When the effect of measurements on the position statistics is taken into account, one finds that the nonlocal dependence on measurements performed on the other component vanishes. Again, we know this must be so, because the predictions must agree with those of quantum mechanics, for which we know that information cannot be transmitted faster than the speed of light.

The nonlocal effects of stochastic mechanics arise basically from the nonlocal nature of the wave function, and the nonlocality "problem" in stochastic mechanics is thus no worse than in quantum mechanics, at least if the wave function is taken seriously. Moreover, if the wave function is not taken seriously, i.e., if realism is abandoned, then it is hard to see the force of the demand for locality, except to ensure consistency with Lorentz invariance, which is presumably ensured in stochastic mechanics, as in quantum mechanics, by the fact that the effects under consideration, whatever their interpretation, convey no information. Furthermore, it is hard to imagine how, as urged by some in discussions of quantum mechanics, the denial of reality, whatever this may mean, can lead to a consistency otherwise absent. At the very least, it would seem that the burden should be, perhaps, on those advocates of traditional quantum mechanics who maintain that nonlocality is unacceptable, to explain why this is so.

10. TOPOLOGICAL EFFECTS

In stochastic mechanics, topological effects arise very simply and naturally. I will illustrate this with two examples, the Aharanov–Bohm effect and quantum statistics.

10.1. The Aharanov–Bohm Effect

Consider a particle moving in the configuration space $Q = \mathbb{R}^3 - \{x^2 + y^2 \leqslant 1\}$ obtained by removing from three-dimensional space the unit vertical (solid) cylinder. What are the *conservative diffusions*, the processes that satisfy the stochastic version of $F = ma$, in this situation, where the configuration space Q is not simply connected? (Strictly speaking, by "conservative diffusions" I mean only those solutions to the stochastic Newton equation or variational equation for which the drift b in the region $\rho > 0$ is locally a gradient. This will automatically be the case for diffusions critical in the sense of Guerra and Morato.[6,13]) It is easy to see, since conservative diffusions are, in fact, defined by local equations, that in addition to the processes governed by a wave function ψ on Q, one also has processes governed by a wave function $\hat{\psi}$ on $\hat{Q}$, the universal covering space of Q, provided that $\hat{\psi}$ at any "level" in $\hat{Q}$ is obtained from $\hat{\psi}$ at the level just below by multiplication by a factor γ, $|\gamma| = 1$. (The wave functions restricted to the various levels thus define the same diffusion process on Q, since multiplication of the wave function by a constant does not affect the corresponding diffusion.) We thus have, given ψ on Q, a one-parameter family of conservative diffusions, labeled by γ.

Consider now sending in a plane wave ψ parallel to the x axis, from $x = -\infty$. Then, for $\gamma = 1$, we obtain an interference pattern for the distribution of particles arriving at a screen on the other side of the cylinder. For $\gamma \neq 1$, we must form a "plane wave" $\hat{\psi}$ on $\hat{Q}$ by setting $\hat{\psi}_k = \gamma^k \psi$, where $\hat{\psi}_k$ is $\hat{\psi}$ restricted to level k. Then, letting $\hat{\psi}$ evolve according to Schrödinger's equation on $\hat{Q}$, we obtain a shifted interference pattern, since components of $\hat{\psi}$ that go around different sides of the cylinder and reach the screen at the same level of $\hat{Q}$ must have originated on different levels, so that when $\gamma \neq 1$, phase relations between these components are shifted.

Now the one-parameter family just described precisely corresponds to the family of solutions obtained by Aharanov and Bohm[21] for the situation in which the cylinder contains a magnetic field B parallel to the z axis (and the charged particle cannot penetrate the cylinder). In this correspondence $\log \gamma \sim i \int dx\, dy\, B$. Aharanov and Bohm wished to emphasize through their analysis the fact that in quantum mechanics the vector potential can produce effects in situations where the particle does not experience any forces. The point I wish to emphasize is that with stochastic mechanics the possibility of the phenomenon predicted by Aharanov and Bohm arises for free motion, with no fields at all—not even the vector potential—in the non-simply-connected region Q. And it seems quite appropriate that the existence of an "inaccessible hole," whose removal renders the topology of the configuration space nontrivial, should give rise to an ambiguity that corresponds to the details of the physics inside the hole, even if this physics classically leaves no trace of itself (i.e., forces) outside the hole.

(One might argue that the same family could be obtained directly in quantum mechanics by allowing multivalued wave functions or by considering the various evolutions corresponding to the different self-adjoint extensions of the Laplacian on nice functions with support inside Q with the half-plane $\{y=0,\ x\geqslant 1\}$ removed. But these possibilities do not appear to be particularly natural, while in stochastic mechanics the question as to whether the corresponding possibilities are natural is not really appropriate; they simply *must* be considered, since they all define conservative diffusions.)

10.2. Statistics[6]

Consider now the configuration space Q of n indistinguishable, non-coincident, particles moving in physical space $\mathbb{R}^3$. The space Q is not simply connected; its universal covering $\hat{Q}$ is $\mathbb{R}^{3n}_{\neq}$, the usual configuration space with coincidences removed. Thus, just as in Section 10.1, conservative diffusions are governed by wave functions $\hat{\psi}$ on $\hat{Q}$, which are such that for configurations related by a permutation π of $\{1,..., n\}$ the corresponding values of $\hat{\psi}$ are related by a factor $\gamma(\pi)$. Moreover, since $\gamma(\pi)$, as is easy to see, must be a character of the permutation group S_n, there are but two possibilities: $\gamma(\pi)=1$, which corresponds to symmetric wave functions, and $\gamma(\pi)=(-1)^{|\pi|}$, corresponding to antisymmetric wave functions. Thus, the set of wave functions $\hat{\psi}$ defining conservative diffusions is not itself a linear space; rather, it is the union of the linear spaces $\mathscr{H}_{\text{symmetric}}$ and $\mathscr{H}_{\text{antisymmetric}}$. In this way, the two familiar possibilities for quantum statistics arise naturally, indeed inevitably, in stochastic mechanics.

The previous two examples illustrate that conservative diffusions on a configuration space Q that is not simply connected are governed by wave functions, on the universal covering space $\hat{Q}$, with "phase relations" given by a character of the fundamental group of Q.

11. ADDITIONAL REMARKS

11.1. Breaking of Unitary Symmetry

Unitary symmetry does not survive the transition from quantum mechanics to stochastic mechanics, in which position has a special status. Since this symmetry is clearly broken, and the primacy of position is so manifest, in the world as we experience it, it is not at all clear that this loss is a defect. (Those who are particularly fond of unitary symmetry should bear in mind that it is respected, of course, by part of stochastic mechanics, namely the purely quantum part.) Nonetheless, it is natural to ask whether something akin to stochastic mechanics can be constructed using observables other then position. This seems to be possible,[32] though processes beyond diffusions will in general be required, since the Hamiltonian will usually not be in "diffusion form" when expressed in other representations. However, these processes, and the theories, the "versions" of stochastic mechanics, of which they will form a part, will in no sense be equivalent to the usual stochastic mechanics I have described, which is based on the position representation.

To clarify this point, it is perhaps worth considering the harmonic oscillator. Here position and momentum play completely symmetric roles, so that a stochastic mechanics based on momentum can be defined just as for position. One then obtains a diffusion process for the momentum, whose time integral should be the position process. But this position process is certainly not the same process as the one arising directly from the usual stochastic mechanics based on position, since the former has differentiable paths and the latter does not. Moreover, this new position process will tend to have a distribution that spreads toward infinity, even for bound states ψ, in disagreement with quantum mechanics. In fact, for the ground state the diffusion process for the momentum is the Ornstein–Uhlenbeck velocity process,[5] which gives rise to a mean-squared displacement for the position that grows linearly for large times.

Thus we see that a stochastic mechanics based on other representations, e.g., momentum, need not be in agreement with quantum mechanics. In view of the unitary symmetry at the level of quantum mechanics itself, this may appear somewhat paradoxical. The reader should, however, bear in mind that a key ingredient in the argument that the predictions of stochastic mechanics agree with those of quantum mechanics is the observation that every measurement is ultimately a position measurement, which is clearly not true for momentum.

11.2. ψ As Generalized Velocity

The role of the wave function ψ in stochastic mechanics is in some sense analogous to that of the velocity in classical mechanics. The introduction of the velocity as a basic dynamical variable was once regarded as counterintuitive[22]: how can there be more to the description of the state at an *instant* beyond a complete description of instantaneous positions? Nonetheless, the inclusion of velocities in the state description leads to a simpler time evolution, in fact a deterministic evolution given by first-order ordinary differential equations. Similarly in stochastic mechanics the inclusion of ψ in the state description leads to (first-order) stochastic differential equations. In classical mechanics the velocity is the extra ingredient necessary to determine the future evolution, while in stochastic mechanics the wave function ψ is required to determine the future stochastically: once ψ is specified, we have a Markov process.

11.3. Generalized Probability Interpretation

Stochastic mechanics can be regarded as generalizing Born's probability interpretation of ψ: Whereas Born tells us that $|\psi(x, t)|^2$ for fixed t is the probability density for the position at time t, stochastic mechanics tells us how to extract from ψ the probability distribution, on path space, for complete trajectories $(x_t)_{t \geqslant 0}$. However, the conceptual details are quite different in the two cases.

For Born $|\psi|^2$ is the distribution, not of an objective, independently existing quantity, but merely of what is "measured." On the other hand, stochastic mechanics provides the distribution of (what it regards as) an objective entity, the trajectory, which, however, is unobservable. Finally, the position at a fixed time is, for stochastic mechanics, both observable and objective.

11.4. The Macroscopic Limit

If we analyze the motion of macroscopic objects using stochastic mechanics, we find[23] that they are undergoing diffusion processes that can be very well approximated by classical mechanics. In particular, the only "component" of the wave function ψ relevant in the macroscopic limit can be represented by a single vector, which we call the velocity.

11.5. Causal Interpretation

Stochastic mechanics, as I have presented it, has a great deal in common with the causal interpretation of quantum mechanics proposed by

David Bohm.[24,25] Here I wish only to mention two important differences. First, Bohm's theory is completely deterministic. Some would regard this as an asset, others a liability. Second, it seems to me that only stochastic mechanics can be regarded as a completely natural generalization of classical mechanics, to processes with "intrinsic randomness." To what natural domain is Bohm's theory a generalization of classical mechanics? (I hope the reader understands why the answer, "to the microscopic," is not satisfactory.)

11.6. The Many-Worlds Interpretation

The advocates of the many-worlds interpretation of quantum mechanics claim that theirs is the only interpretation consistent with Schrödinger's equation alone, without additional (measurement) postulates. In this interpretation, "measurements" cause the universe to "branch" into as many copies as the wave function has "components." Apart from its bizarre character, this interpretation suffers from two very serious defects: (i) the "branching" should depend upon a choice of basis, and (ii) its derivation of the probability interpretation of $|\psi|^2$ in situations in which the various "outcomes" are not equally likely involves the circularity of asserting that small $\|\cdot\|^2$ means small "fraction of the worlds." To overcome these difficulties, position coordinates would have to be regarded as special, and an interpretive assumption would have to be adopted: that the "number of worlds" is proportional to $|\text{amplitude}|^2$. This is already close to stochastic mechanics. In fact, if one takes branching seriously in the many-worlds interpretation, i.e., if one requires the theory to identify the evolution of a single world through time and branches, then we might as well simply say that the position coordinates are undergoing a stochastic process whose marginal single-time distributions are given by $|\psi|^2$. Then stochastic mechanics provides a natural description of the full process, which, however, is not completely determined by the single-time marginals alone. [If one does not take the branching seriously, the many-worlds interpretation becomes much stranger still, because then the notion of the same observer (over time) is an illusion—it just *seems* that we belong to a consistent history because memory has this feature (consistency) built in.]

11.7. Reversibility

Just as Newtonian (or Einsteinian) mechanics is strictly reversible ($t \to -t$, $v \to -v$ maps solutions to solutions), so is stochastic mechanics ($t \to -t$, $\psi \to \psi^*$ maps conservative diffusions to conservative diffusions).

(This is not true of all formulations of stochastic mechanics, but it is true of the one adopted by Nelson, which is based on a symmetric definition of stochastic acceleration, and is the one I have been discussing. Much, if not all, of what I have said prior to this subsection would apply as well to the other versions.) It might be argued that quantum mechanics is also strictly reversible, but this is true only if we identify quantum mechanics completely with Schrödinger's equation, ignoring measurement and collapse.

11.8. Agreement with Experience

Great care must be taken when trying to decide whether a theory "agrees with experience." I will illustrate this for stochastic mechanics with a simple example.

Consider a macroscopic "particle" moving (in one dimension) between two walls, at 0 and at L, with completely sharp, large (macroscopic) energy E_n (n large). Then $\psi_n \sim \sin(n\pi x/L)$, which has nodes that are extremely close together. Since the particle cannot diffuse across these nodes, it would appear to be at rest on the macroscopic scale, despite the fact that it has large energy. Should we regard this state of affairs as conflicting with (macroscopic) experience?

Even if the preceding analysis were correct, there would be no conflict, since what we would experience is a particle apparently at rest, which we would regard as having zero energy, and there is nothing that contradicts experience in this. But the analysis is not correct, because in order to decide how the situation would *appear*, we must take into account the effect of looking at the particle, of illuminating it. And when we do this we find, as in the analysis of the measurement process in Section 7, that once the particle is observed, the effective wave function should be collapsed to a small wave packet; more precisely, ψ_n should be replaced by ψ_n multiplied by a modulating factor that vanishes outside of a small interval. But this effective wave function is a superposition of a left-moving and a right-moving wave packet, and after these wave packets separate the particle will be in the support of one of them, with which it will continue to move, with a macroscopic velocity corresponding to the energy E_n. So what would, in fact, be *observed* is precisely what should be observed for a particle with energy E_n. (Of course the particle is under continuous "observation" through interaction with photons, cosmic rays, etc., so that the "collapse" just described is always occurring, and the original wave function ψ_n is nonphysical.)

11.9. Exceptional Conservative Diffusions

In some special situations there exist conservative diffusions that do not correspond to wave functions satisfying Schrödinger's equation. For example, the conservative diffusions corresponding to the excited energy levels of the hydrogen atom, which have nodal surfaces, can be decomposed into conservative diffusions supported in the connected components separated by the nodal surfaces. However, presumably the only conservative diffusions that are stable under small perturbations are those given by the "canonical formula," i.e., that arise in the canonical way from solutions to Schrödinger's equation. The situation here is analogous to that in statistical mechanics: though in certain situations there are stationary states other than those given by the Gibbs formula, only the Gibbs states are stable.[26,27]

11.10. Stochastic Completion

The stochastic mechanics I have described could be called the *stochastic completion* of nonrelativistic, spinless quantum mechanics. Stochastic completions of other quantum theories have been obtained, incorporating spin,[20] and including some field theories.[28] Of course, in constructing the stochastic completion of a theory, a decision must be made as to which variables are to undergo the stochastic process, but this may somehow be forced.

It is, of course, not necessary that every quantum theory have a (satisfactory) stochastic completion, only those to be taken seriously. [Thus, it would not be so bad if $P(\phi)_2$ had no satisfactory stochastic completion.] It is presumably much too early to speculate about the stochastic completion of superstring theory.

11.11. Wave–Particle Duality

In stochastic mechanics the origin of wave–particle duality is very clear. There is no need to invoke complementarity to preserve consistency.

12. CONCLUSION

Stochastic mechanics is both the natural generalization of classical mechanics to processes with intrinsic randomness, as well as a prescription for extracting from quantum mechanics, i.e., from the wave function, a detailed description of what the particles comprising a quantum system are doing between measurements.

What does stochastic mechanics add to quantum mechanics? It asserts that, in fact, the (position) coordinates actually exist and are doing something, even when unobserved. Moreover, what they are doing is very natural; the evolution of coordinates is described by a diffusion process that pops out when one looks for a generalization of $F = ma$ (or the Lagrangian variational principle) to systems with intrinsic randomness. At the same time, in so obtaining this diffusion process, Schrödinger's equation itself naturally arises.

Is there anything to object to in this? Well, has not Heisenberg (or Bohr) shown, by taking into account the irreducible disturbances arising from the existence of the quantum of action, that simultaneous position and momentum, and hence a detailed description of what is happening on the microscopic level between observations, is meaningless? But such reasoning, which has always been unsatisfying to many because of its positivistic, antirealistic flavor (not to speak of its using classical concepts —colliding billiard balls, etc.—to draw radical conclusions about a non-classical realm) must surely be fallacious; it claims to establish the impossibility of just the sort of detailed description provided, in a very natural way, by stochastic mechanics. [Concerning the Heisenberg argument, it might be asserted that due to sensitive dependence on initial conditions—as well as the very large number of degrees of freedom involved—detailed prediction of trajectories for particles in a classical gas is (in principle) impossible. Why does this assertion then not invalidate classical statistical mechanics, based as it is on classical mechanics? But, in fact, the situation is quite the opposite: it is the chaotic character of the motion that has traditionally been used to justify the methods of statistical mechanics.[(29)]]

It is true that we may not be able to check, through observation, the details of the description provided by stochastic mechanics, but this is quite another matter from asserting the impossibility of any such description, not to speak of denying the possibility that something is actually going on. (A positivist would presumably deny these distinctions.) Moreover, if one maintains that there can be no advantage in appending to quantum mechanics some unobservable elements, one need only recall the great many conceptual difficulties arising from quantum mechanics, all of which, I have argued, are dissolved by stochastic mechanics. [The paradoxes, in particular, can be regarded as a manifestation of an unbridgable gap between pure quantum mechanics and familiar (classical) reality. Stochastic mechanics bridges this gap in the simplest way possible, by incorporating the classical description, i.e., the coordinates.] This would seem a strong argument indeed for accepting an unobservable reality, which perhaps might never have been denied had the theory of stochastic processes been sufficiently well-developed at the time quantum mechanics was conceived.

It is true that the detailed description provided by stochastic mechanics is indeterministic. But it is generally accepted that with quantum mechanics we are already confronted with the fact that the laws of nature are indeterministic. Moreover, the emergence of randomness in the context of pure quantum mechanics is a mystery, while in stochastic mechanics it is obvious. Indeed, stochastic mechanics may be regarded as the theory that most naturally arises from the assumption that nature is intrinsically random.

I will close on a conservative note. It is often claimed[(30,31)] by supporters of its traditional interpretation that quantum mechanics marks a radical change in our world view, requiring, in fact, the abandonment of "realism"—the view that there is an objective reality, at least partially apprehended through scientific investigation. But if this theory does indeed necessitate profound changes in epistemology and metaphysics, then evidence of the strongest kind must be advanced on its behalf. The stranger or more radical (i.e., the smaller the *a priori* probability of) the proposal, the firmer must be the evidence.

Now regardless of whether we accept stochastic mechanics as the correct description of microscopic reality, its very existence calls into question the validity of the radical claims of the traditionalists. How can quantum mechanics be more likely than a theory far less radical? The more the traditional supporter of quantum mechanics emphasizes how truly revolutionary quantum mechanics is, how profound a change it requires in our epistemology and our view of the objectivity of the (microscopic) world, the less sustainable that position becomes. [The same remarks apply (in spades) to those who advocate a (nonclassical) quantum logic interpretation of quantum mechanics.]

It would seem more prudent to say that quantum mechanics (without philosophical additions) is a provisional theory that works remarkably well and that presumably reflects important features of what is "really" going on, features that will be more fully appreciated in terms of, if not stochastic mechanics, a theory yet to be propounded.

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